Introduction to Abstract Algebra
with Applications to Social Systems

Women in Mathematics
SWIM
Summer Workshop in Mathematics
FOR HIGH SCHOOL STUDENTS
Princeton

Course II
Lecture
Notes
5 of 7

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The DeGroot Model

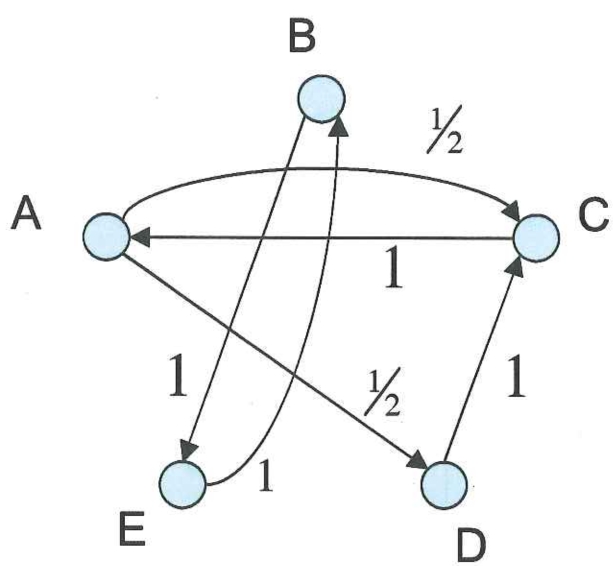
Corollary

Corollary. A consensus is reached in the DeGroot model if and only if the group is *strongly connected* and *aperiodic*.

The DeGroot Model

Example 4

Suppose there are 5 individuals with the following influence network



$$T = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The DeGroot Model

Example 4

We cannot find a stationary distribution for the full network, but we can rearrange the rows and columns to reveal 2 closed, disjoint subsets:

$$T = \begin{array}{c} \\ A \\ C \\ D \\ B \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & C & D & B & E \end{array} \\ \left(\begin{array}{ccc|cc} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

The DeGroot Model

Example 4

$$\pi T = \pi$$

$$T_{A,C,D} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_{B,E} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

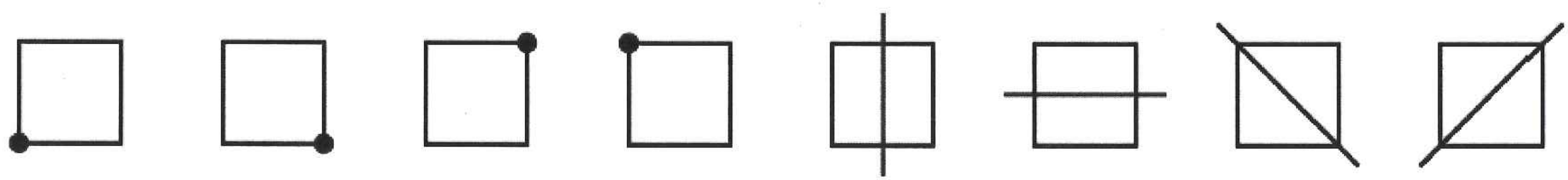
$$\pi_{A,C,D} = \left(2/5, \quad 2/5, \quad 1/5\right)$$

$$\pi_{B,E} = \left(1/2, \quad 1/2\right)$$

Symmetries of the Square

Description

Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. Describe all possible ways in which this can be done.

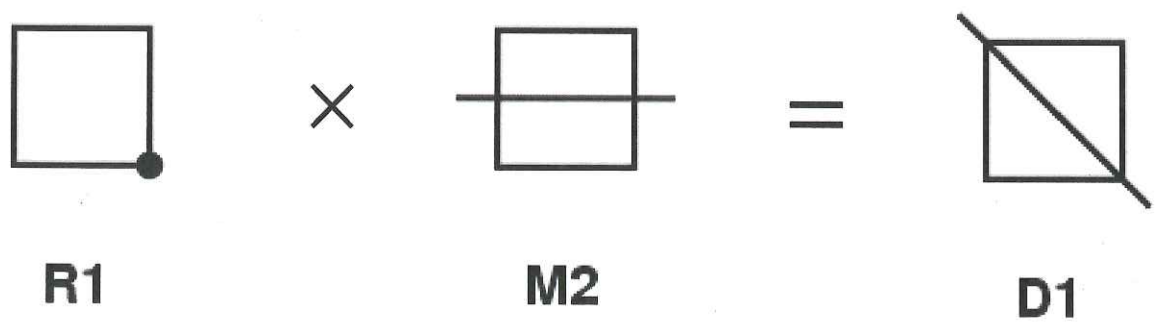


R0 R1 R2 R3 M1 M2 D1 D2

Symmetries of the Square

Composition

Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. Describe all possible ways in which this can be done.



Symmetries of the Square

Cayley Table

	R0	R1	R2	R3	M1	M2	D1	D2
R0	R0	R1	R2	R3	M1	M2	D1	D2
R1	R1	R2	R3	R0	D1	D2	M2	M1
R2	R2	R3	R0	R1	M2	M1	D2	D1
R3	R3	R0	R1	R2	D2	D1	M1	M2
M1	M1	D2	M2	D1	R0	R2	R3	R1
M2	M2	D1	M1	D2	R2	R0	R1	R3
D1	D1	M1	D2	M2	R1	R3	R0	R2
D2	D2	M2	D1	M1	R3	R1	R2	R0

Group

Definition

Let *G* be a nonempty set together with a binary operation (usually called multiplication). We say *G* is a *group* under this operation if the following properties are satisfied:

1. <u>Closure</u>: For all *a*, *b* in *G*, *ab* is also in *G*.
2. <u>Identity</u>: There is an element in *G* (called the *identity*) such that *ae* = *ea* = *a* for all *a* in *G*.
3. <u>Inverses</u>: For each element *a* in *G*, there is an element *b* in *G* (called an *inverse* of a) such that *ab* = *ba* = *e*.
4. <u>Associativity</u>: The operation is associative; that is, (*ab*)*c* = *a*(*bc*) for all *a*, *b*, *c* in *G*.

Symmetries of the Square

Cayley Table

	R0	R1	R2	R3	M1	M2	D1	D2
R0	R0	R1	R2	R3	M1	M2	D1	D2
R1	R1	R2	R3	R0	D1	D2	M2	M1
R2	R2	R3	R0	R1	M2	M1	D2	D1
R3	R3	R0	R1	R2	D2	D1	M1	M2
M1	M1	D2	M2	D1	R0	R2	R3	R1
M2	M2	D1	M1	D2	R2	R0	R1	R3
D1	D1	M1	D2	M2	R1	R3	R0	R2
D2	D2	M2	D1	M1	R3	R1	R2	R0

→ NOT Commutative (Abelian)

Groups

Abelian

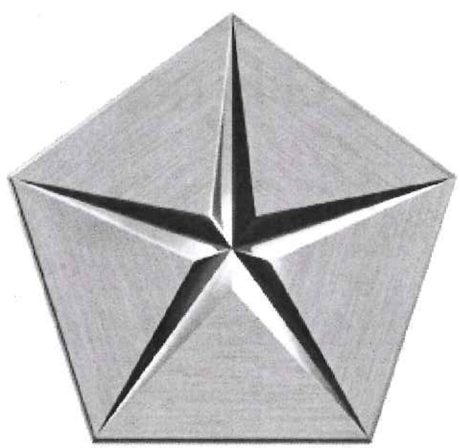
Definition: A group G is Abelian (i.e., commutative) if for all a, b in G, $ab = ba$. Otherwise, G is non-Abelian.

Groups

Dihedral Groups

Symmetries can be computed for any regular *n*-gon $(n \geq 3)$.

The corresponding group is denoted by D_n and is called the *dihedral group of order 2n*.



D_5

D_3

Groups

Example 1

The set of integers under ordinary addition is a group.

1. Closure
2. Identity: 0
3. Inverses: The inverse of *a* is -*a*.
4. Associativity

Groups

Example 2

The set of integers under ordinary multiplication is NOT a group.

1. Closure
2. Identity: 1
3. Inverses: No inverses. (For example: there is no integer b such that $5b = 1$.)
4. Associativity

Groups

Example 3

The subset {1, -1, *i*, -*i* } of the complex numbers under complex multiplication is a group.

1. Closure
2. Identity: 1
3. Inverses: -1 is its own inverse. The inverse of *i* is -*i* and the inverse of -*i* is *i*.
4. Associativity

Groups

Example 4

The set of all 2 x 2 matrices with real entries under componentwise addition is a group.

1. Closure
2. Identity: The identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
3. Inverses: The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.
4. Associativity

Groups

Example 5

The set of all 2 x 2 matrices with real entries under matrix multiplication is NOT a group.

1. Closure
2. Identity: The identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
3. Inverses: The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

Inverses do not exist when the determinant is 0.

Properties of Groups

Uniqueness of the Identity

Theorem: In a group G, there is only one identity element.

Proof. Suppose there are two identities of G, namely e and e'. Then

1. $ae = a$ for all a in G, and
2. $e'a = a$ for all a in G.

Let $a = e'$ in (1) and $a = e$ in (2).

Then $e'e = e'$ and $e'e = e$.

Therefore, $e = e'$.

□

Properties of Groups

Cancellation

Theorem: In a group G, the right and left cancellation laws hold; that is, $ba = ca$ implies $b = c$, and $ab = ac$ implies $b = c$.

Proof. Suppose $ba = ca$. Let a' be an inverse of a. Multiplying on the right by a' gives $(ba)a' = (ca)a'$. Associativity yields $b(aa') = c(aa')$. Then, $be = ce$. Therefore, $b = c$.

Similarly, one can prove that $ab = ac$ implies $b = c$ by multiplying by a' on the left. □

Properties of Groups

Uniqueness of Inverses

Theorem: For each element a in a group G, there is a unique element b in G such that $ab = ba = e$.

Proof. Suppose b and c are both inverses of a. Then $ab = e$ and $ac = e$. Then, $ab = ac$. Therefore, by cancellation, $b = c$.

□

Properties of Groups

Order

Definition: The number of elements of a group (finite or infinite) is called its *order*. We will use $|G|$ to denote the order of G.

Definition: The *order* of an element g in a group G is the smallest positive integer n such that $g^n = e$. (In additive notation, this would be $ng = 0$.) If no such integer exists, we say that g has *infinite order*. The order of an element g is denoted by $|g|$.

Properties of Groups

Subgroup

Definition: If a subset *H* of a group *G* is itself a group under the operation of *G*, we say that *H* is a subgroup of *G*.

Symmetries of the Square

Cayley Table

	R0	R1	R2	R3	M1	M2	D1	D2
R0	R0	R1	R2	R3	M1	M2	D1	D2
R1	R1	R2	R3	R0	D1	D2	M2	M1
R2	R2	R3	R0	R1	M2	M1	D2	D1
R3	R3	R0	R1	R2	D2	D1	M1	M2
M1	M1	D2	M2	D1	R0	R2	R3	R1
M2	M2	D1	M1	D2	R2	R0	R1	R3
D1	D1	M1	D2	M2	R1	R3	R0	R2
D2	D2	M2	D1	M1	R3	R1	R2	R0

Primitive Societies

Marriage

- Each individual has a marriage type.
- Only individuals of the same marriage type may marry.
- Marriage type is determined by parents' type and gender.
- Brother-sister marriages are not allowed.
- Some first-cousin marriages are allowed.

Primitive Marriage Rules

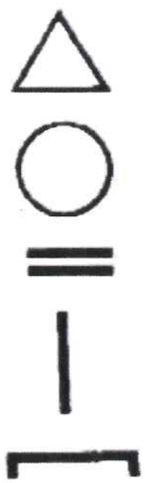
Research Questions

- What is the structure underlying primitive marriage rules?
- What types of marriages are allowed?
- What types of marriages are not allowed?

Family Tree Diagrams

First Generation

Legend:



Male

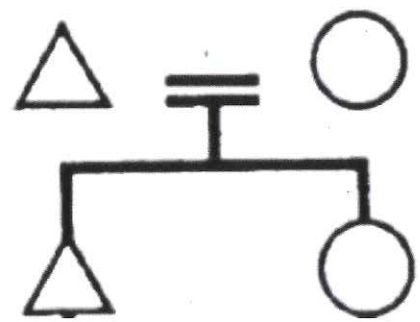
Female

Marriage

Descendent

Sibling

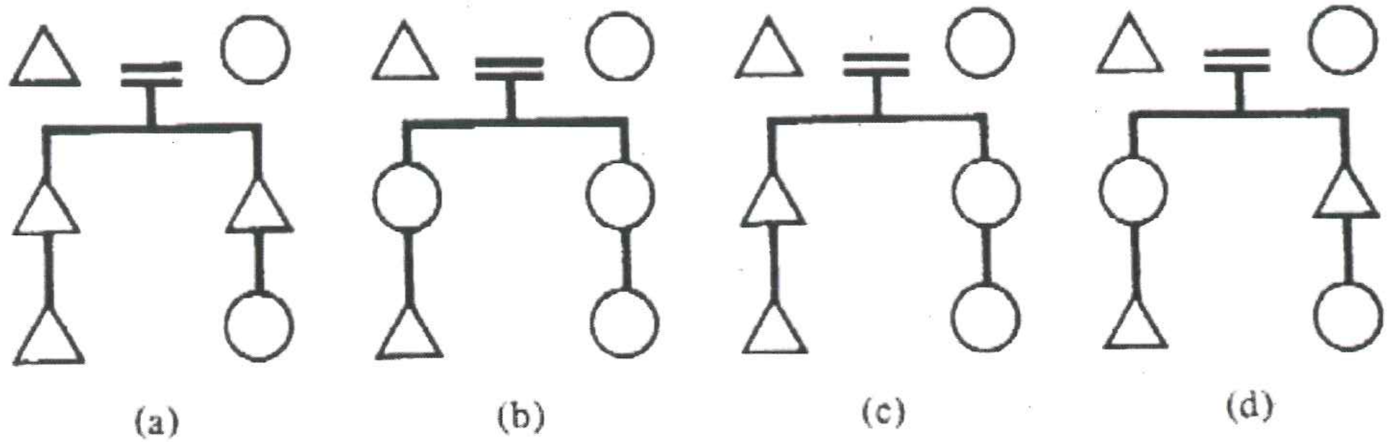
Brother-sister relationship



Family Tree Diagrams

Second Generation

Four types of first-cousin relationships

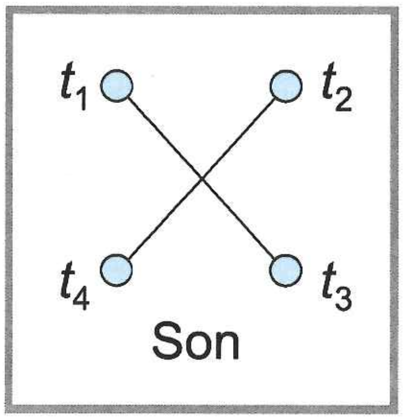


Primitive Marriage Rules

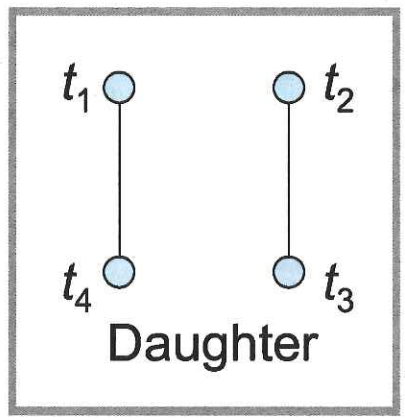
Example - *Kariera* Society, First Generation

Table 1. *Kariera* Marriage Types – First Generation

Parents	t_1	t_2	t_3	t_4
Son	t_3	t_4	t_1	t_2
Daughter	t_4	t_3	t_2	t_1



$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



$$D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

NOTE: *S* and *D* are **permutation matrices**

Primitive Marriage Rules

Example - *Kariera* Society, First Generation

Vector of Marriage Types $t = (t_1, \quad t_2, \quad t_3, \quad t_4)$

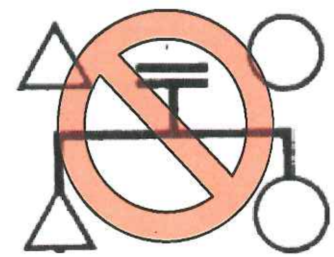
Son

$$(t_1, \quad t_2, \quad t_3, \quad t_4)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = (t_3, \quad t_4, \quad t_1, \quad t_2)$$

Daughter

$$(t_1, \quad t_2, \quad t_3, \quad t_4)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = (t_4, \quad t_3, \quad t_2, \quad t_1)$$

$tS \neq tD$



Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

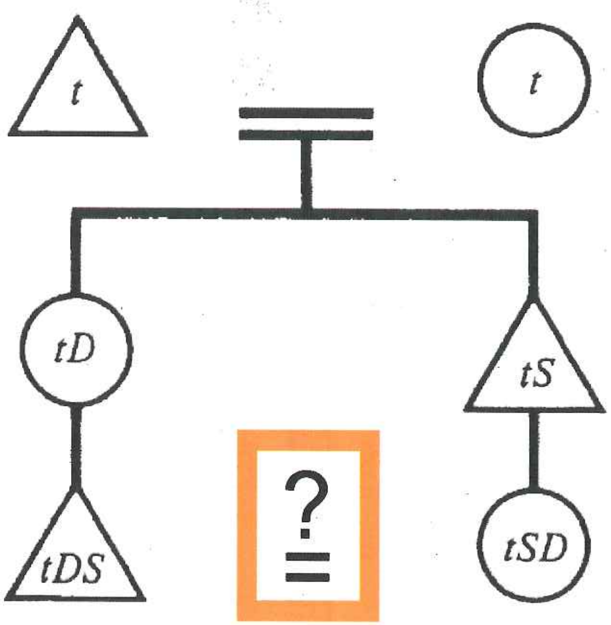
Table 2. *Kariera* Marriage Types – Second Generation

Parents	t_1	t_2	t_3	t_4
Son	t_3	t_4	t_1	t_2
Son's Son	t_1	t_2	t_3	t_4
Son's Daughter	t_2	t_1	t_4	t_3
Daughter	t_4	t_3	t_2	t_1
Daughter's Son	t_2	t_1	t_4	t_3
Daughter's Daughter	t_1	t_2	t_3	t_4

Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

First-cousin relationship (d)



Permutation Matrices

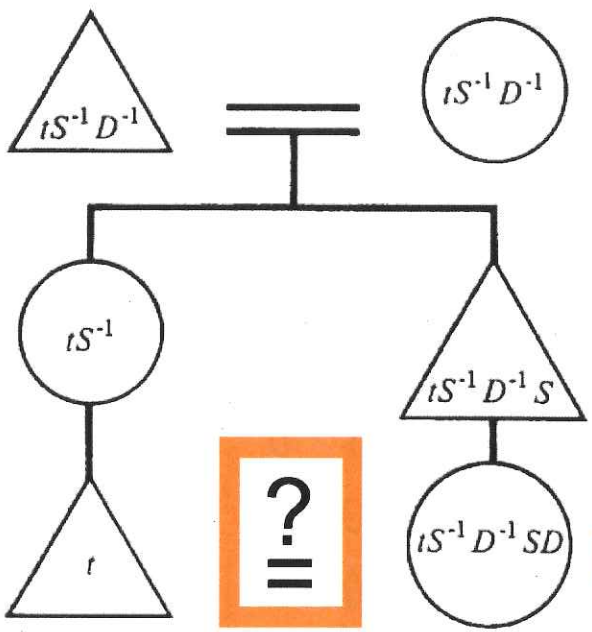
$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$DS = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = SD$$

Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

First-cousin relationship (d)



Permutation Matrices

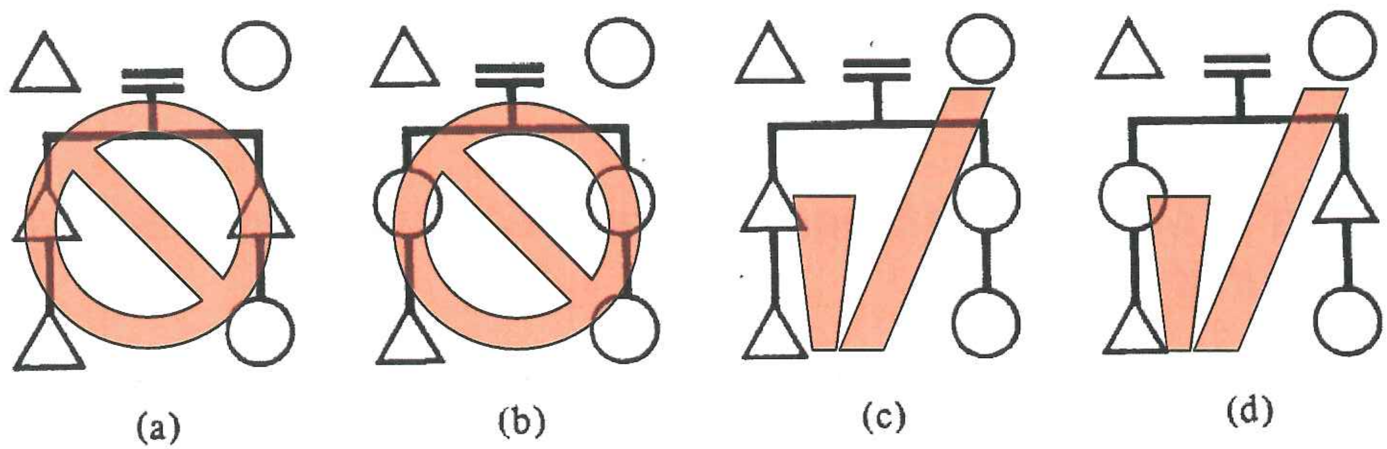
$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$M = S^{-1}D^{-1}SD = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

Four types of first-cousin relationships



Primitive Marriage Rules

Example - *Kariera* Society, Second Generation

$$G = \{I, \quad S, \quad D, \quad SD\}$$

Theorem 1: *In the group generated by S and D, every element except I is a complete permutation.*

Theorem 2: *Marriage between relatives of a given kind is always permitted if M = I.*

Theorem 3: *Marriage between relatives of a given kind is never permitted if M is a complete permutation.*

Theorem 4: *$S^{-1}D$ (brother-sister relation) is a complete permutation.*